On the similarity solutions for a steady MHD equation

Jean-David Hoernel

Department of Mathematics, Technion-Israel Institute of Technology, Amado Bld., 32000 Haifa, Israel

Received 5 September 2006; accepted 27 October 2006

Abstract

In this paper, we investigate the similarity solutions for the steady laminar incompressible boundary layer equations governing the magnetohydrodynamic (MHD) flow near the forward stagnation point of two-dimensional and axisymmetric bodies. This leads to the study of a boundary value problem involving a third order autonomous ordinary differential equation. Our main results are the existence, uniqueness and non-existence for concave or convex solutions.

© 2006 Elsevier B.V. All rights reserved.

PACS: 02.60.Lj; 47.15.Cb; 47.65.d

Keywords: Boundary layer; Similarity solution; Third order non-linear differential equation; Boundary value problem; MHD

1. Introduction

Boundary layer flow of an electrically conducting fluid over moving surfaces emerges in a large variety of industrial and technological applications. It has been investigated by many researchers, Wu [1] has studied the effects of suction or injection on a steady two-dimensional MHD boundary layer flow on a flat plate, Takhar et al. [2] studied a MHD asymmetric flow over a semi-infinite moving surface and numerically obtained the solutions. An analysis of heat and mass transfer characteristics in an electrically conducting fluid over a linearly stretching sheet with variable wall temperature was investigated by Vajravelu and Rollins [3]. Muhapatra and Gupta [4] treated the steady two-dimensional stagnation-point flow of an incompressible viscous electrically conducting fluid towards a stretching surface, the flow being permeated by a uniform transverse magnetic field. For more details see also [5–8] and the references therein. Motivated by the above works, we aim here to give analytical results about the third order non-linear autonomous differential equation

\[ f''' + \frac{m+1}{2}ff'' + m(1-f'^2) + M(1-f') = 0 \quad \text{on } [0, \infty) \]  

accompanied by the boundary conditions

\[ f(0) = \alpha, \quad f'(0) = \beta, \quad f'(\infty) = 1, \]  

E-mail address: j-d.hoernel@wanadoo.fr

1007-5704/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.cnsns.2006.10.010

Please cite this article in press as: Hoernel J-D, On the similarity solutions for a steady MHD equation, Commun Nonlinear Sci Numer Simul (2007), doi:10.1016/j.cnsns.2006.10.010
where \( x, \beta, m, M \in \mathbb{R} \) and \( f'(\infty) := \lim_{t \to \infty} f'(t) \). Eq. (1) is very interesting because it contains many known equations as particular cases. Let us give a few examples.

Setting \( M = 0 \) in (1), leads to the well-known Falkner–Skan equation (see [9,10] and the references therein). While the case \( M = -m \) reduces (1) to equation that arises when considering the mixed convection in a fluid saturated porous medium near a semi-infinite vertical flat plate with prescribed temperature studied by many authors, we refer the reader to [11–13] and the references therein. The case \( M = m = 0 \) is refereed to the Blasius equation introduced in [14] and studied by several authors (see for example [15–17]).

Recently some results have been obtained by Brighi and Hoernel [18], about the more general equation

\[
f'''' + ffi'''' + g(f') = 0 \quad \text{on } [0, \infty)
\]

with the boundary conditions

\[
f(0) = \alpha, \quad f'(0) = \beta, \quad f'(\infty) = \lambda,
\]

where \( \alpha, \beta, \lambda \in \mathbb{R} \) and \( g \) is a given function.

Guided by the analysis of [18] we shall prove that problem (1), (2) admits a unique concave or a unique convex solution for \( m > -1 \) according to the values of \( M \). We give also non-existence results for \( m \in \mathbb{R} \) and related values of \( M \).

2. Flow analysis

Let us suppose that an electrically conducting fluid (with electrical conductivity \( \sigma \)) in the presence of a transverse magnetic field \( B(x) \) is flowing past a flat plate stretched with a power-law velocity. According to \([19] \) and \([20] \), such phenomenon is described by the following equations:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e u_x + v \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B^2(x)}{\rho} (u_e - u).
\]

Here, the induced magnetic field is neglected. In a cartesian system of coordinates \((O, x, y)\), the variables \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions respectively. We will denote by \( u_e(x) = cx^m, c > 0 \) the external velocity, \( B(x) = B_0 x^{-\frac{a}{b}} \) the applied magnetic field, \( m \) the power-law velocity exponent, \( \rho \) the fluid density and \( v \) the kinematic viscosity.

The boundary conditions for problem (5), (6) are

\[
u(x, 0) = u_e(x) = ax^m, \quad v(x, 0) = v_e(x) = bx^{-\frac{a}{b}}, \quad u(x, \infty) = u_e(x),
\]

where \( u_e(x) \) and \( v_e(x) \) are the stretching and the suction (or injection) velocity respectively and \( a, b \) are constants. Recall that \( a > 0 \) is referred to the suction, \( a < 0 \) to the injection and \( a = 0 \) to the impermeable plate.

A little inspection shows that Eqs. (5) and (6) accompanied by conditions (7) admit a similarity solution. Therefore, we introduce the dimensional stream function \( \psi \) in the usual way to get the following equation:

\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = u_e u_x + v \frac{\partial^3 \psi}{\partial y^3} + \frac{\sigma B^2(x)}{\rho} (u_e - u).
\]

The boundary conditions become

\[
\frac{\partial \psi}{\partial y}(x, 0) = ax^m, \quad \frac{\partial \psi}{\partial x}(x, 0) = -bx^{-\frac{a}{b}}, \quad \frac{\partial \psi}{\partial y}(x, \infty) = cx^m.
\]

Defining the similarity variables as follows:

\[
\psi(x, y) = x^{\frac{a}{b}+1} f(t) \sqrt{v} c \quad \text{and} \quad t = x^{\frac{a}{b}+1} y \sqrt{v}/c
\]

and substituting in Eqs. (8) and (9) we get the following boundary value problem

Please cite this article in press as: Hoernel J-D, On the similarity solutions for a steady MHD equation, Commun Nonlinear Sci Numer Simul (2007), doi:10.1016/j.cnsns.2006.10.010
\[
\begin{align*}
\begin{cases}
f'''' + \frac{m+1}{2} ff''' + m(1-f'^2) + M(1-f') = 0, \\
f(0) = \alpha, \quad f'(0) = \beta, \quad f'(\infty) = 1,
\end{cases}
\end{align*}
\]
where \( \alpha = \frac{2b}{(m+1)\sqrt{c}} \), \( \beta = \frac{b}{c} \) and \( M = \frac{\sigma_b^2}{c\rho} > 0 \) is the Hartmann number and the prime is for differentiating with respect to \( t \).

### 3. Various results

First, we notice that

**Remark 1.** Let \( \beta = 1 \), then the function \( f(t) = t + \alpha \) is a solution of the problem (1) and (2) for any values of \( m \) and \( M \) in \( \mathbb{R} \).

We cannot say much about the uniqueness of the previous solution, but if \( g \) is another solution with \( gg''(0) > 0 \) then, since \( g'(0) = g'(\infty) = 1 \) there exists \( t_0 > 0 \) such that \( g'(t_0) > 1 \), \( gg''(t_0) = 0 \) and \( g'''(t_0) \leq 0 \). However, from (1) we obtain that for \( m > 0 \) and \( M > 0 \), \( g'''(t_0) = -m(1-g'^2(t_0)) - M(1-g'(t_0)) > 0 \) and thus a contradiction.

Suppose that \( f \) verifies Eq. (1) only. We will now establish some estimations for the possible extrema of \( f' \).

**Proposition 3.1.** Let \( f \) be a solution of Eq. (1) and \( t_0 \) be a minimum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \geq 0 \)), if it exists. For such a point \( t_0 \) we have the following possibilities, according to the values of \( m \) and \( M \).

- For \( m < 0 \)
  - if \( m < -2m \), then \(-1 - \frac{M}{m} \leq f'(t_0) \leq 1\),
  - if \( M = -2m \), then \( f'(t_0) = 1 \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \leq -1 - \frac{M}{m} \).
- For \( m = 0 \)
  - if \( M < 0 \), then \( f'(t_0) \leq 1 \),
  - if \( M > 0 \), then \( 1 \leq f'(t_0) \).
- For \( m > 0 \)
  - if \( M < -2m \), then \( f'(t_0) \leq 1 \) or \( -1 - \frac{M}{m} \leq f'(t_0) \),
  - if \( M > -2m \), then \( 1 \leq f'(t_0) \) or \( f'(t_0) \leq -1 - \frac{M}{m} \).

**Proof.** Let \( t_0 \) be a minimum of \( f' \) with \( f \) a solution of (1). Using Eq. (1) and the fact that \( f''''(t_0) = 0 \), we obtain that
\[
f'''(t_0) + m(1-f'^2(t_0)) + M(1-f'(t_0)) = 0.
\]
Setting \( p(x) = m(1-x^2) + M(1-x) \), we have that \( f'''(t_0) \geq 0 \) leads to \( g(f'(t_0)) \leq 0 \) and the results follows.

Let us remark that in both cases \( m = M = 0 \) and \( m > 0 \), \( M = -2m \) we cannot deduce anything about \( f'(t_0) \).

**Proposition 3.2.** Let \( f \) be a solution of Eq. (1) and \( t_0 \) be a maximum for \( f' \) (i.e. \( f''(t_0) = 0 \) and \( f'''(t_0) \leq 0 \)), if it exists. For such a point \( t_0 \) we have the following possibilities, according to the values of \( m \) and \( M \).

- For \( m < 0 \)
  - if \( m < -2m \), then \( f'(t_0) \leq -1 - \frac{M}{m} \) or \( f'(t_0) \geq 1 \),
  - if \( M > -2m \), then \( f'(t_0) \leq 1 \) or \( f'(t_0) \geq -1 - \frac{M}{m} \).
- For \( m = 0 \)
  - if \( M < 0 \), then \( f'(t_0) \leq 1 \),
  - if \( M > 0 \), then \( f'(t_0) \leq 1 \).
• For $m > 0$
  - if $M < -2m$, then $1 \leq f'(t_0) \leq -1 - \frac{m}{m'}$.
  - if $M = -2m$, then $f'(t_0) = 1$.
  - if $M > -2m$, then $-1 - \frac{m}{m'} \leq f'(t_0) \leq 1$.

Proof. We proceed as in the previous Proposition, but, this time, with the condition $g(f'(t_0)) \geq 0$. Let us remark that in both cases $m < 0$, $M = -2m$ and $m = M = 0$ we cannot deduce anything about $f'(t_0)$.

We will now use the two previous Propositions to deduce results about the possible extremas for $f'$ with $f$ a solution of the problem (1) and (2).

Theorem 1. Let $f$ be a solution of the problem (1) and (2), $t_0$ be a minimum for $f'$ (i.e. $f''(t_0) = 0$ and $f'''(t_0) \geq 0$), if it exists, and $t_1$ be a maximum for $f'$ (i.e. $f''(t_1) = 0$ and $f'''(t_1) \leq 0$), if it exists. For such points $t_0$ and $t_1$, we have the following possibilities for the values of $f'$.

• For $m < 0$
  - if $M < -2m$, then $-1 - \frac{m}{m'} \leq f'(t_0) \leq 1 \leq f'(t_1)$.
  - if $M = -2m$, then $f'(t_0) = 1$.
  - if $M > -2m$, then $1 \leq f'(t_0) \leq -1 - \frac{m}{m'} \leq f'(t_1)$.

• For $m = 0$
  - if $M < 0$, then $f'(t_0) \leq 1 \leq f'(t_1)$.
  - if $M > 0$, then $f'$ cannot vanish.

• For $m > 0$
  - if $M < -2m$, then $f'(t_0) \leq 1 \leq f'(t_1) \leq -1 - \frac{m}{m}$.
  - if $M = -2m$, then $f'(t_1) = 1$.
  - if $M > -2m$, then $f'(t_0) \leq -1 - \frac{m}{m'} \leq f'(t_1) \leq 1$.

Proof. Taking into account the fact that $f'(t) \to \infty$ as $t \to \infty$, combining Propositions 3.1 and 3.2 leads to the results.

Remark 2. A consequence of the previous Theorem is that, for $m = 0$ and $M > 0$ all the solutions of the problem (1) and (2) have to be concave or convex everywhere.

4. The concave and convex solutions

In the following section we will first prove that, under some hypotheses, the problem (1) and (2) admits a unique concave solution or a unique convex solution for $m > -1$. Then, we will give some non-existence results about the concave or convex solutions for $m \in \mathbb{R}$ according to the values of $M$.

To this aim, we will use the fact that, if $f$ is a solution of the problem (1) and (2), then the function $h$ defined by

$$f(t) = \sqrt{\frac{2}{m+1}} h\left(\sqrt{\frac{m+1}{2}} t\right)$$

with $m > -1$, is a solution of the equation

$$h'' + hh' + g(h') = 0$$

on $[0, \infty)$, with the boundary conditions

Please cite this article in press as: Hoernel J-D, On the similarity solutions for a steady MHD equation, Commun Nonlinear Sci Numer Simul (2007), doi:10.1016/j.cnsns.2006.10.010
We get that the problem (12) and (13) admits a unique concave solution by (11) and that verifies (12) and (13). Then, as

\begin{equation}
\left. \begin{array}{c}
h(0) = \sqrt{\frac{m+1}{2}} \alpha, \\
h'(0) = \beta, \\
h'(\infty) = 1,
\end{array} \right\} \quad \text{(13)}
\end{equation}

and where

\begin{equation}
g(x) = \frac{2m}{m+1} (1-x^2) + \frac{2M}{m+1} (1-x).
\end{equation}

In the remainder of this section we will made intensive use of the results established in the paper [18] by Brighi and Hoernel.

**Remark 3.** It is immediate that for any \( \alpha \in \mathbb{R} \), if \( \beta < 1 \) there is no concave solutions of the problem (1), (2) and if \( \beta > 1 \) there is no convex solutions of the problem (1) and (2).

### 4.1. Concave solutions

Let us begin with the two following results about existence, uniqueness and non-existence of concave or convex solutions for the problem (1) and (2).

**Theorem 2.** Let \( \alpha \in \mathbb{R} \) and \( \beta > 1 \). Then, there exists a unique concave solution of the problem (1), (2) in the two following cases

- \(-1 < m \leq 0 \) and \( M > -m(\beta + 1) \),
- \( m > 0 \) and \( M \geq -2m \).

Moreover, there exists \( \alpha < 1 < \sqrt{\alpha^2 + 4 \frac{\beta^2}{m+1}} \) such that \( \lim_{t \to \infty} \{ f(t) - (t + 1) \} = 0 \) and for all \( t \geq 0 \) we have \( t + \alpha \leq f(t) \leq t + 1 \).

**Proof.** Let \( f \) be a solution of the problem (1) and (2) with \( m > -1 \) and consider the function \( h \) that is defined by (11) and that verifies (12) and (13). Then, as \( g(1) = 0 \) for the function \( g \) defined by (14), using Theorem 1 of [18] we get that the problem (12) and (13) admits a unique concave solution \( h \) for every \( \alpha \in \mathbb{R} \) and \( \beta > 1 \) if and only if \( g(x) < 0 \) for all \( x \) in \( (1, \beta) \). Noticing that \( g(-\frac{M}{m}-1) = 0 \), the previous condition is verified if \( m \geq 0 \) and \( -\frac{M}{m} - 1 \leq 1 \), if \( m = 0 \) and \( M > 0 \) or if \(-1 < m < 0 \) and \( M > -2m \) and \( \beta < -\frac{M}{m} - 1 \). Using now Proposition 1 of [18] for \( h \) we have the second result. \( \square \)

**Theorem 3.** Let \( \beta > 1 \). Then, there are no concave solutions of the problem (1) and (2) in the following cases

- \( \alpha \in \mathbb{R} \), \( m \leq -1 \) and \( M \leq -2m \),
- \( \alpha \leq 0 \), \( -1 < m < -\frac{1}{2} \) and \( M \leq -\frac{5m+1}{2} \),
- \( \alpha \leq 0 \), \( m = -\frac{1}{2} \) and \( M < \frac{1}{2} \),
- \( \alpha < 0 \), \( m = -\frac{1}{2} \) and \( M = \frac{1}{2} \),
- \( \alpha \leq 0 \), \( m > -\frac{1}{2} \) and \( M \leq -\frac{(3m+1)\beta+2m}{2} \).

**Proof.** Let \( \alpha \in \mathbb{R} \), \( m \leq -1 \) and \( f \) be a concave solution of the problem (1) and (2). We then have that \( f' > 1 \), \( f'' < 0 \), \( f''' > 0 \) everywhere and \( f(t) > 0 \) for \( t \) large enough because \( f'(t) \to 1 \) as \( t \to \infty \). Using the fact that \( \frac{m+1}{2} f'' > 0 \) near infinity, we obtain from (1) that

\[ f''' \leq -m(1-f'^2) - M(1-f') \]

near infinity. As the polynomial function \( -m(1-x^2) - M(1-x) \) is negative for all \( x \) in \( [1, \infty) \) if \( m \leq -1 \) and \( M \leq -2m \), we get that \( f''' < 0 \) near infinity because \( f' > 1 \) everywhere. This is a contradiction, so concave solutions cannot exist in this case. Consider now \( m > -1 \) and \( h \) a solution of the problem (12) and (13). Let us define the function \( \hat{g} \) by \( \hat{g}(x) = g(x) - x^2 + x \), a simple calculation leads to

Please cite this article in press as: Hoernel J-D. On the similarity solutions for a steady MHD equation, Commun Nonlinear Sci Numer Simul (2007), doi:10.1016/j.cnsns.2006.10.010
\[ \hat{g}(x) = \frac{1}{m+1} (-3m+1)x^2 + (m+1-2M)x + 2(m+M). \]

Then, the Theorem 2 from [18] tells us that problem (12) and (13) admits no concave solutions for \( x \leq 0 \) if \( \forall x \in [1, \beta], \hat{g}(x) \geq 0 \) and \( -x + \max_{x \in [1, \beta]} g(x) > 0. \) These conditions leads to the results for problem (1) and (2) with \( m > -1. \]

The results from Theorems 2 and 3 are summarized in the Fig. 1 in which the plane \((m, M)\) contains three disjoints regions A, B and C that corresponds to

- A: Existence of a unique concave solution for \( m > -1, \beta > 1 \) and \( x \in \mathbb{R}, \)
- B: No concave solutions for \( m > -1, \beta > 1 \) and \( x \leq 0, \)
- C: No concave solutions for \( m \leq -1, \beta > 1 \) and \( x \in \mathbb{R}. \)

4.2. Convex solutions

We will now give existence, uniqueness and non-existence results for the convex solutions of the problem (1) and (2).

**Theorem 4.** Let \( x \in \mathbb{R} \) and \( 0 \leq \beta < 1. \) Then, there exists a unique convex solution of the problem (1) and (2) in the following cases:

- \(-1 < m < 0 \) and \( M \geq -2m, \)
- \( m \geq 0 \) and \( M > -m(\beta + 1). \)

Moreover, there exists \( l > x \) such that \( \lim_{t \to -\infty} \{ f(t) - (t + l) \} = 0 \) and for all \( t \geq 0 \) we have \( t + x \leq f(t) \leq t + l. \)

**Proof.** We proceed the same way as for Theorem 2, but with the condition that \( g(x) > 0 \) for all \( x \in [\beta, 1). \) We conclude by using first the Theorem 3 from [18], then the Proposition 2 from [18]. \( \square \)

**Theorem 5.** Let \( 0 \leq \beta < 1. \) Then, there are no convex solutions of the problem (1) and (2) in the following cases:

- \( x \in \mathbb{R}, m \leq -1 \) and \( M \leq -m(\beta + 1), \)
- \( x \leq 0, -1 < m < -\frac{1}{3} \) and \( M \leq -\frac{(3m+1)x+2m}{2}, \)
- \( x \leq 0, m = -\frac{1}{3} \) and \( M < \frac{1}{3}. \)

Fig. 1. Existence and non-existence for concave solutions.
\[ a < 0, \ m = -\frac{1}{2} \text{ and } M = \frac{1}{3},
\]
\[ a \leq 0, \ m > -\frac{1}{2} \text{ and } M \leq -\frac{5m+1}{2}.
\]

**Proof.** For \( m > -1 \) and \( a \leq 0 \), the proof is the same as the previous one, but this time we need that \( \forall x \in [\beta, 1], \)
\[ \hat{g}(x) \leq 0 \text{ and } -x + \max_{x \in [\beta, 1]} \hat{g}(x) > 0, \]
according to the Theorem 4 from [18].

Consider now \( m \leq -1, \ a \in \mathbb{R} \) and let \( f \) be a convex solution of the problem (1) and (2). We have that \( \beta \leq f' < 1, \ f'' > 0, \ f''' < 0 \) everywhere and that \( f(t) > 0 \) for \( t \) large enough because \( f'(t) \rightarrow 1 \) as \( t \rightarrow \infty \). According to Eq. (1), we have that
\[ f''' = -\frac{m+1}{2} ff'' - m(1 - f'^2) - M(1 - f') \]
with \( -\frac{m+1}{2} ff'' > 0 \) near infinity. As the polynomial function \( -m(1-x^2) - M(1-x) \) is positive for all \( x \) in \( [\beta, 1] \) if \( m \leq -1 \) and \( M \leq -m(\beta + 1) \), we get that \( f''' > 0 \) near infinity because \( b \leq f' < 1 \). This is a contradiction, thus convex solutions cannot exist in this case. \( \square \)

The results from Theorems 4 and 5 are summarized in the Fig. 2 in which the plane \((m, M)\) contains three disjoint regions A, B and C that corresponds to

- **A:** Existence of a unique convex solution for \( m > -1, \ 0 \leq \beta < 1 \) and \( a \in \mathbb{R} \),
- **B:** No convex solutions for \( m > -1, \ 0 \leq \beta < 1 \) and \( a \leq 0 \),
- **C:** No convex solutions for \( m \leq -1, \ 0 \leq \beta < 1 \) and \( a \in \mathbb{R} \).

### 5. Conclusion

In this paper, we have shown the existence of a unique concave or a unique convex solution of the problem (1) and (2) for \( m > -1 \), according to the values of \( M \). We also have obtained non-existence results for \( m \in \mathbb{R} \) and related values of \( M \), as well as some clues about the possible behavior of \( f' \). It is a first work on this problem, there is still much left to do because of its complexity.

Notice that the case \( M = -2m \) plays a particule role in the problem (1) and (2), because it is the only one for which we are able to predict the possible changes of concavity for \( f \). Its study will be the subject of a forthcoming paper.

### Acknowledgements

The author thanks Prof. B. Brighi for his many advices and for introducing him to the similarity solutions family of problems, Dr. Z. Hammouch for bringing the subject of this paper to his attention and the Depart-
We wish to express our gratitude to The Department of Mathematics of the Technion for supporting his researches through a Postdoctoral Fellowship in the frame of the RTN “Fronts-Singularities”.

References